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Asymptotic behavior of solutions to a differential equation with state-dependent delay☆

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ABSTRACT

In this paper, we investigate the asymptotic behavior of solutions to a differential equation with state-dependent delay. It is shown that every bounded solution of such an equation tends to a constant as $t \to \infty$. Our results improve and extend some corresponding ones already known.

1. Introduction

In this paper, we are concerned with the following differential equation with state-dependent delay

$$x'(t) = -F(x(t)) + G(x(t-r)), \qquad r = r(x(t-\delta)) \tag{1.1}$$

where $r : \mathbf{R}^1 \to (0, \infty)$ is a continuous function, $\delta > 0$ is a constant, $F, G \in C(R^1)$ and either $G(x) \geq F(x)$ for all $x \in R^1$ or $G(x) \leq F(x)$ for all $x \in R^1$.

A special case of Eq. (1.1) of the form

$$x'(t) = -F(x(t)) + F(x(t-r)), \qquad r = r(x(t-\delta)), \tag{1.2}$$

with $r \equiv 1$ appears in several applications (see [1,2]). When r is a constant, Eq. (1.2) has been extensively studied (see, for example, [2,3], and the references cited therein) because of their applications in modelling population growth, the spread of epidemics, the dynamics of capital stocks and so on. Over the past several years it has become apparent that equations with state-dependent delay arise in several areas such as in classical electrodynamics [4], in population models [5], in models of commodity price fluctuations [6], and in models of blood cell productions [7]. Recently, it was shown in J. Terjéki and M. Bartha [8] that each bounded solution of Eq. (1.2) tends to a constant as $t \to \infty$ under the assumption that F is strictly increasing.

Now, a corresponding question arises: can we show every bounded solution of Eq. (1.1) tends to a constant as $t \to \infty$ provided F is nondecreasing on R^1. The main purpose of this paper is to discuss the asymptotic behavior of bounded solutions of Eq. (1.1). By using monotonicity arguments in this work, we show that, assuming that F is nondecreasing on R^1 and some

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additional conditions hold, every bounded solution of Eq. (1.1) is convergent to a constant. Our approach is quite different from those of [3,8] and our results are new.

The paper is organized as follows. In Section 2, we introduce some necessary notations and establish some preliminary results, which are important in the proofs of our main results. Based on the preparations in Section 2, we state and prove our main results in Section 3.

2. Preliminary results

In this section, some important properties of Eq. (1.1) will be presented. These are of importance in proving our main results in Section 3.

Let $\tilde{C} = C((-\infty, 0], \mathbf{R}^1)$. We say that a function $x : \mathbf{R}^1 \to \mathbf{R}^1$ is a solution of Eq. (1.1) through $\varphi \in \tilde{C}$ if x is continuous on $\mathbf{R}^1$, x is differentiable on $(0, \infty)$, it satisfies Eq. (1.1) for all $t \geq 0$, and $x|_{(-\infty,0]} = \varphi$. Here, $x'(0)$ denotes the right-hand derivative of x at 0.

Now, we consider the following differential equation with state-dependent delay

$$x'(t) = -F(x(t)) + G(x(t - \bar{r})), \qquad \bar{r} = \bar{r}(x(t - \delta)), \tag{2.1}$$

where $\bar{r} : \mathbf{R}^1 \to (0, \infty)$ is a bounded continuous function, and

$$\tau = \max\{\delta, \sup_{x \in \mathbf{R}^1} \bar{r}(x)\} \geq \inf_{x \in \mathbf{R}^1} \bar{r}(x) > 0.$$

Let $C = C([-\tau, 0], \mathbf{R}^1)$ be the phase space. We say that a function $x : [-\tau, \infty) \to \mathbf{R}^1$ is a solution of Eq. (2.1) through $\varphi \in C$ if x is continuous on $[-\tau, \infty)$, x is differentiable on $(0, \infty)$, it satisfies Eq. (2.1) for all $t \geq 0$, and $x|_{[-\tau,0]} = \varphi$. Again $x'(0)$ denotes the right-hand derivative of x at 0.

As pointed out in [8], all solutions $x(t)$ of Eq. (1.1) do not tend to constants when $t \to \infty$, since Eq. (1.1) may have unbounded solutions. We will show that the bounded solutions $x(t)$ of Eq. (1.1) converge to constants as $t \to \infty$. Let $x(t)$ be a bounded solution of Eq. (1.1); then there exist constants $A > 0$ and $M > 0$ such that

$$-A \leq x(t) \leq A, \quad M = \sup_{-A \leq x \leq A} r(x) \geq \inf_{-A \leq x \leq A} r(x) > 0.$$

Set

$$\bar{r}(x) = \begin{cases} r(A), & x \geq A, \\ r(x), & -A < x < A, \\ r(-A), & x \leq -A, \end{cases} \tag{2.2}$$

and

$$\tau = \max\{\delta, M\} > 0.$$

Then, $x(t)$ is a bounded solution of Eq. (2.1) with initial data

$$\varphi(t) = x(t)|_{[-\tau,\,0]} \in C = C([-\tau, 0], \mathbf{R}^1).$$

Therefore, to prove that the bounded solutions $x(t)$ of Eq. (1.1) converge to constants as $t \to \infty$, it suffices to show that the bounded solutions $x(t)$ of Eq. (2.1) converge to constants as $t \to \infty$.

In what follows, $\mathbf{R}^1_+$ denotes the set of all nonnegative real numbers. Let $C([-\tau, 0], \mathbf{R}^1)$ be equipped with the supremum norm. Then, $C = C([-\tau, 0], \mathbf{R}^1)$ is a Banach space. Define $C_+ = C([-\tau, 0], \mathbf{R}^1_+)$. It follows that C_+ is an order cone in the Banach space C and hence C_+ induces a closed partial ordered relation on C. For any $\varphi, \psi \in C$, we write: $\varphi \leq \psi$ if $\psi - \varphi \in C_+$; $\varphi < \psi$ if $\varphi \leq \psi$ and $\varphi \neq \psi$; $\varphi \ll \psi$ if $\psi - \varphi \in \operatorname{Int} C_+$. For any $A \subseteq C$, we write: $\varphi \leq A$ if $\varphi \leq \psi$ for all $\psi \in A$; $\varphi < A$ if $\varphi < \psi$ for all $\psi \in A$, $\varphi \ll A$ if $\varphi \ll \psi$ for all $\psi \in A$. Similarly, we can define "$\geq$", "$>$" and "$\gg$". For instance, $\varphi \geq \psi$ if $\varphi - \psi \in C_+$. If $\sigma \geq 0$ and $x \in C([-\tau, \sigma], R^1)$, then, for any $t \in [0, \sigma]$, $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$. For any $\alpha \in \mathbf{R}^1$, we define $\widehat{\alpha} \in C$ by $\widehat{\alpha}(\theta) = \alpha$, $\theta \in [-\tau, 0]$. Moreover, for $\varphi \in C$, we use $x_t(\varphi)$ $(x(t, \varphi))$ to denote the solution of (2.1) with the initial data $x_0(\varphi) = \varphi$. Also, $x_t(\varphi, F)$ $(x(t, \varphi, F))$ denotes the solution of the equation

$$x'(t) = -F(x(t)) + F(x(t - \bar{r})), \quad \bar{r} = \bar{r}(x(t - \delta)) \tag{2.3}$$

with the initial data $x_0(\varphi, F) = \varphi$.

Lemma 2.1 ([9]). *Let $0 < T \in \mathbf{R}^1$ be given and $d \in C([t_0, t_0 + T], \mathbf{R}^1)$. Then, for any constant x_0, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + d(t), \\ x(t_0) = x_0 \end{cases} \tag{2.4}$$

has a unique solution $x(t)$ on $[t_0, t_0 + T]$.

Lemma 2.2. *Let $\varphi \in C$. Then $x_t(\varphi)$ exists and is unique on $\mathbf{R}^1_+$.*

Proof. Let

$$d(t) = G(x(t - \bar{r}(x(t - \delta, \varphi)), \varphi)) = G(\varphi(t - \bar{r}(\varphi(t - \delta)))), \quad t \in [0, \underline{\delta}],$$

where $\underline{\delta} = \min\{\delta, \inf_{x\in\mathbf{R}^1} \bar{r}(x)\}$. Consider the solution $x(t)$ of the following initial value problem,

$$\begin{cases} x'(t) = -F(x(t)) + d(t), \\ x(0) = \varphi(0). \end{cases}$$

By Lemma 2.1, $x(t)$ exists and is unique on $[0, \underline{\delta}]$, that is, $x_t(\varphi)$ exists and is unique on $[0, \underline{\delta}]$. Then $x_t(\varphi)$ exists and is unique on $\mathbf{R}^1_+$ by induction. This completes the proof. □

Let

$$\Phi : \mathbf{R}^1_+ \times C \longrightarrow C. \Phi(t, \varphi) = x_t(\varphi).$$

It follows from Lemma 2.2 and Theorem 2.2 in [10] that Φ is a continuous map. Then Φ is a continuous semi-flow on C. To our further discussion, we introduce the following notations for the sake of convenience.

For $\varphi \in C$, define $O(\varphi) = \{x_t(\varphi) \in C : t \geq 0\}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in C, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$. If $O(\varphi)$ is bounded, define

$$\omega(\varphi) = \bigcap_{t\geq 0} \overline{O(x_t(\varphi))},$$

i.e., $\omega(\varphi) = \{\psi \in C :$ there exists a sequence $t_k \to +\infty$ such that $x_{t_k}(\varphi) \to \psi\}$. It easy to check that $\omega(\varphi)$ is nonempty, compact, invariant and connected.

We introduce the following assumptions:

(A_+) $G \geq F$, and for any $\alpha \in \mathbf{R}^1$, there exist $\varepsilon > 0$ and $L > 0 \in \mathbf{R}^1$ such that $F(x) - F(\alpha) \leq L(x - \alpha)$ for all $x \in [\alpha, \alpha + \varepsilon]$.
(A_-) $G \leq F$, and for any $\alpha \in \mathbf{R}^1$, there exist $\varepsilon > 0$ and $L > 0 \in \mathbf{R}^1$ such that $F(x) - F(\alpha) \geq L(x - \alpha)$ for all $x \in [\alpha - \varepsilon, \alpha]$.

Lemma 2.3. *Assume (A_+) holds, $\varphi \in C$, and $\alpha \in \mathbf{R}^1$ such that $\varphi \geq \widehat{\alpha}$. Then, $x_t(\varphi) \geq \widehat{\alpha}$ for $t \geq 0$. Moreover, either $x_t(\varphi) \gg \widehat{\alpha}$ or $x_t(\varphi) = \widehat{\alpha}$ for all $t \geq 2\tau$.*

Proof. It follows from Proposition 1.1 of Smith [11] that

$$x_t(\varphi) \geq x_t(\widehat{\alpha}, F) = \widehat{\alpha} \quad \text{for } t \geq 0.$$

Now, we shall consider two cases as follows:

Case i. $x(t, \varphi) = \alpha$ for all $t \in [0, 2\tau]$. From (2.1) we have

$$-F(x(t)) + G(x(t - \bar{r}(x(t - \delta, \varphi)), \varphi)) = 0, \quad \text{for all } t \in [\tau, 2\tau],$$

which implies that $-F(\alpha) + G(\alpha) = 0$. Thus, $x(t, \varphi) = \widehat{\alpha}$ for all $t \geq 2\tau$. Therefore, $x_t(\varphi) = \widehat{\alpha}$ for all $t \geq 2\tau$.

Case ii. $x(t, \varphi) > \alpha$ for some $t_1 \in [0, 2\tau]$. We will next prove that $x(t, \varphi) > \alpha$ for all $t \in [t_1, +\infty)$. Otherwise,

$$t_2 = \inf\{t \geq t_1 : x(t, \varphi) = \alpha\} < +\infty.$$

In view of (A_+), there exist constants $\eta \in (0, t_2 - t_1)$ and $L > 0$ such that

$$\begin{aligned} x'(t, \varphi) &= -F(x(t, \varphi)) + G(x(t - \bar{r}(x(t - \delta, \varphi)), \varphi)) \\ &\geq -F(x(t, \varphi)) + F(x(t - \bar{r}(x(t - \delta, \varphi)), \varphi)) \\ &\geq -F(x(t, \varphi)) + F(\alpha) \\ &\geq -L(x(t, \varphi) - \alpha), \end{aligned} \tag{2.5}$$

where $t \in [t_2 - \eta, t_2]$. Thus,

$$x(t_2, \varphi) \geq \alpha + (x(t_2 - \eta, \varphi) - \alpha)e^{-L\eta} > \alpha,$$

a contradiction to the definition of t_2. It follows that $x(t, \varphi) > \alpha$ for all $t \in [t_1, +\infty)$. Hence, $x_t(\varphi) \gg \widehat{\alpha}$ for all $t \geq 2\tau$. This completes the proof. □

Similarly, we can prove the following result.

Lemma 2.4. *Assume (A_-) holds, $\varphi \in C$, and $\alpha \in \mathbf{R}^1$ such that $\varphi \leq \widehat{\alpha}$. Then, $x_t(\varphi) \leq \widehat{\alpha}$ for $t \geq 0$. Moreover, either $x_t(\varphi) \ll \widehat{\alpha}$ or $x_t(\varphi) = \widehat{\alpha}$ for all $t \geq 2\tau$.*

3. Main results

With the preparations in Section 2, we are ready to state and prove our main results.

Theorem 3.1. *Assume (A_+) holds, and $\varphi \in C$. If $O(\varphi)$ is bounded, then there exists $\alpha^* \in R^1$ such that $\omega(\varphi) = \{\widehat{\alpha}^*\}$.*

Proof. Let $\alpha^* = \sup\{\alpha \in R^1 : \widehat{\alpha} \le \omega(\varphi)\}$. Since $\omega(\varphi)$ is compact, we obtain $\alpha^* \in R^1$. We will show that $\omega(\varphi) = \{\widehat{\alpha}^*\}$. Otherwise, $\omega(\varphi) \setminus \{\widehat{\alpha}^*\} \ne \phi$. According to the invariance of $\omega(\varphi)$, we have $x_{2\tau}(\omega(\varphi)) = \omega(\varphi)$. It follows that

$$x_{2\tau}(\omega(\varphi)) \setminus \{\widehat{\alpha}^*\} \ne \phi,$$

and hence there exists $\psi \in \omega(\varphi)$ such that

$$x_{2\tau}(\psi) > \widehat{\alpha}^*.$$

From Lemma 2.3 and the fact that $\psi \ge \widehat{\alpha}^*$, we obtain

$$x_{2\tau}(\psi) \gg \widehat{\alpha}^*.$$

Therefore, there exists $\alpha^{**} > \alpha^*$ such that

$$x_{2\tau}(\psi) \gg \widehat{\alpha}^{**}.$$

Again by the invariance of $\omega(\varphi)$ and its definition, there exists $t_3 > 0$ such that

$$x_{t_3}(\varphi) \ge \widehat{\alpha}^{**} \gg \widehat{\alpha}^*.$$

By Lemma 2.3, we get

$$x_t(x_{t_3}(\varphi)) \ge \widehat{\alpha}^{**} \gg \widehat{\alpha}^* \quad \text{for } t \ge 0.$$

Thus,

$$\omega(\varphi) \ge \widehat{\alpha}^{**} \gg \widehat{\alpha}^*.$$

This contradicts the definition of α^*. The proof of the theorem is now complete. □

Theorem 3.2. *Assume* (A_-) *holds, and* $\varphi \in C$. *If* $O(\varphi)$ *is bounded, then there exists* $\alpha^* \in R^1$ *such that* $\omega(\varphi) = \{\widehat{\alpha}^*\}$.

Proof. By a similar argument to that in the proof of Theorem 3.1, the conclusion of Theorem 3.2 follows immediately by applying Lemma 2.4. □

Putting Theorems 3.1 and 3.2 together, we obtain the following result.

Corollary 3.1. *Let* (A_+) *and* (A_-) *hold and* $\varphi \in C$. *Then there exists* $\alpha^* \in R^1$ *such that* $\omega(\varphi) = \{\widehat{\alpha}^*\}$.

Proof. From Lemmas 2.3 and 2.4, it follows that $O(\varphi)$ is bounded. Therefore, by Theorem 3.1 or 3.2, the conclusion of Corollary 3.1 holds. □

Remark 3.1. If (A_+) (or (A_-)) holds, then by Theorems 3.1 and 3.2, each bounded solution of (1.1) tends to a constant as $t \longrightarrow +\infty$. Since F is nondecreasing on R^1, our results are new. Moreover, Our approach is quite different from those of [3,8,10,11].

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